OPTIMAL DESIGN OF RANDOM KNOCKOUT TOURNAMENTS

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Chapter 1: Introduction

1.1 PREAMBLE

Operations research and optimization models have been proposed for many years in order to design sports tournaments, displaying different desirable properties and satisfying various feasibility constraints. Indeed, sports competitions may involve several logistical and economic issues, which has led many researchers to take an interest in the topic. Sport's contribution to the economy is constantly growing and, according to the European Commission's study, based in 2012 data, the share of sport-related Gross Domestic Product (GDP) in the EU is 2.12 % and amounts to € 279.7 bn (The house of european sport, 2018).

Some of the properties, or performance indicators, of tournaments have also been examined from a more theoretical point of view. Indeed, different types of competition have been studied in the context of sports and other domains. Although sport is the first sector that comes to mind when speaking about tournaments, another key field is politics—the organization of elections for example. Nevertheless, during the elaboration of the themes of this thesis, sports tournaments will be the focus.

With sport and tournaments being a high-profile and popular domain, many optimization problems have already been analyzed and studied in the literature. These studies have aimed in particular at the most familiar and highly mediatized sports, such as football played under the auspices of the Union of European Football Associations (UEFA) and the Fédération Internationale de Football Association (FIFA), (see Green (2015) or Lasek et al. (2013)) or basketball organized by the National Collegiate Athletic Association (NCAA) (see Schwertman (1991) or Khatibi (2015)). However, no specific sport will be analyzed in this thesis, as it will only focus on the optimal design of sport tournament.

"Designing an optimal contest is both a matter of significant financial concern for the organizers, participating individuals, and teams, and a matter of consuming personal interest for millions of fans" (Szymanski, 2010, p. 1137).

Before moving to the research question, the word "tournament" needs to be defined.

Definition 1. *"**Tournaments** are used to select a single winner from a group of participants in a sporting event or a paired-comparison experiment" (Horen & Riezman, 1985, p1).*

Tournaments offer a model of the statistical method called the method of paired comparisons or pairwise comparisons—that is, comparing two elements with each other in order to identify the preferred option. This technique of paired comparisons is used when several objects have to be considered on the basis of various criteria or measures.

In the context of sport, at each stage of the tournament, every paired comparison will be called a match. It can thus be said that a tournament is a rule that indicates how the teams or players will be compared in order to choose the winner of a competition with a set of p players. This rule will, consequently, generate a sequence of games to be played (Maurer, 1975).

Although many studies and articles have examined the design of sports tournaments, few articles deal with tournament structure. Indeed, there are many different ways to build a tournament, depending on various criteria, which is explained in detail in Chapter 3.

1.2 RESEARCH QUESTION AND GOALS

The starting point of this thesis was the article by Alder et al. (2017) on random knockout tournaments. In this paper, the authors demonstrated that, in the special case where only one player has a higher relative strength than the others, the design that maximizes the strongest player's winning probability is the balanced structure. The authors also assumed, without any proof, that in the more general case where players have different strengths, the balanced structure still maximizes the strongest player's winning probability.

Consequently, the main objective of this thesis will be to examine this assumption by answering the following question: "In a knockout tournament, that is to say a direct elimination tournament, what type of structure optimizes the strongest player's probability of winning?"

During the elaboration of this thesis, different sports tournaments and their specific terminology were defined, winning probabilities of random knockout tournaments were computed, and an algorithm was developed in order to provide indications of the effectiveness of the tournament structure and to evaluate and draw conclusions regarding the types of structure to be chosen. Indeed, the algorithm will allow to see how the interaction between the player strength and the chosen tournament structure affects its probability of winning.

In order to be able to infer which type of structure maximizes or minimizes the strongest player's probability of winning, all possible structures were analyzed. To do this, tournaments were considered as full unordered binary trees. Once these structures were obtained, a simulation was performed to randomly assign strengths to the different players, and they were then randomly placed in the tournament bracket. Finally, the winning probability of each player for each round was computed in order to identify the highest one and thus to obtain the optimal structure for the strongest player.

For ease of reading, terms specific to sports tournaments and binary trees are explained in section 1.3 and 3.1.

1.3 TERMINOLOGY

Explanations of the particular terms used in this thesis are given in the following subsections. Common terms such as tournament type, structure, and format are sometimes misused or can have different meanings depending on the author. The purpose of this section is to define their meanings and set the direction in which the terms will be used.

These various terms are frequently used in sports literature, which is why no basic references are mentioned.

1.3.1 Tournament format

According to Adler et al. (2017), the tournament format is the number of matches played in each round. A round refers to the set of matches that can be played at the same time. For example, Figure 1 shows a three-round tournament.

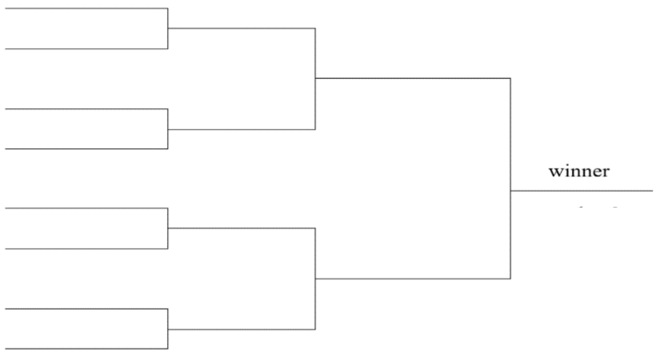

Figure 1: A three-round tournament

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A tournament is composed of R rounds, with round i consisting of m_i matches. The total number of matches will therefore be equal to $\sum_{i=1}^{R} m_i = p - 1$ (Adler et al., 2017), with p being the number of players or teams. As can be observed in Figure 1, the first round is composed of four matches, the second round has two matches, and the last round, the final, has one match. The winner of this last match will be declared the winner of the tournament. This constitution of matches will thus refer to the tournament format 4 – 2 – 1.

1.3.2 Tournament type

Depending on the sport, many different types of competition may exist. Indeed, there are many different systems for organizing tournaments, each with its own advantages and drawbacks. The three most common are round-robin, single elimination, and double-elimination tournaments.

a) The most studied tournament in the statistical literature is the **(single) round-robin** tournament, in which players or teams play once against each other. The winner is determined by the number of victories or by the total number of points accumulated during the games played.

As this type of tournament requires many matches, it is only valuable when the number of players or teams is small, and when games are played quickly. Figure 2 presents an example of a round robin tournament involving eight teams.

Team	Wins	Losses
1.		
2.		
3.		
4.		
5.		
6.		
7.		
8.		

Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
2 vs 1	3 vs 4	6 vs 2	7 vs 5	1 vs 3	4 vs 5	7 vs 3
3 vs 8	1 vs 7	7 vs 8	8 vs 4	4 vs 2	8 vs 1	8 vs 2
4 vs 7	8 vs 6	4 vs 1	2 vs 3	5 vs 8	2 vs 7	1 vs 5
5 vs 6	2 vs 5	5 vs 3	6 vs 1	6 vs 7	3 vs 6	6 vs 4

Figure 2: Round-robin bracket with eight participants. Retrieved from https://www.printyourbrackets.com/

b) As can be seen in Figure 4, a **knockout tournament**, also known as a **single-elimination tournament**, is an event in which the participants—individual players or teams—are placed into a bracket with L leaves. Knockout tournaments differ from round-robin tournaments in that not all possible pairs of games occur. The draw is, therefore, of great importance here.

The draw corresponds to the way players are placed in the bracket, either (1) randomly, or (2) in a predefined way according to the players' strengths.

i) The first case is termed a random knockout tournament. The p players are randomly paired and then randomly planted into the L leaves of the tournament bracket.

ii) The second case is termed seeding. The seeding corresponds to the assignment of players to the L leaves according to their strengths (Vu et al., 2009). In the most used seeded knockout tournament, players or teams are placed in the bracket in a manner that ensures the two best players or teams to not meet until the last round of the competition (i.e. the final). This typical seeding, called the standard seeding, is shown at Figure 3 for an eight player competition. The numbering represents the strength of the player, the strongest player being player 1 and the weakest, in this case, player 8.

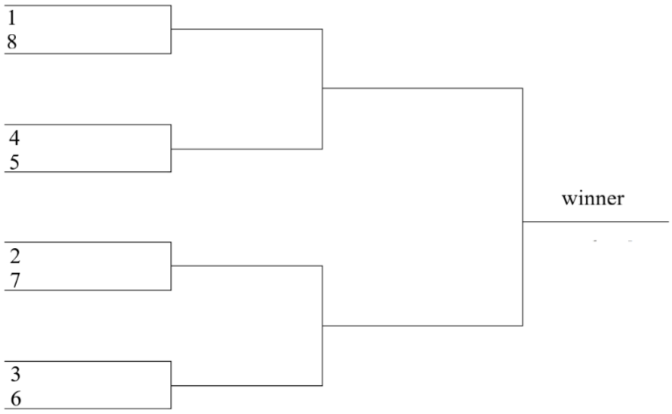

Figure 3: Standard seeding in a knockout tournament with eight players

Single-elimination tournaments are widespread in sports competitions. They are regularly used as a qualifying tournament when the number of teams is high, or as a playoff tournament—that is, a series of games in which teams compete at the end of a sports season to determine who is eligible to play in a higher division in the following season (Karpov, 2016).

Using the knockout tournament type is the simplest way to organize a tournament, as players compete two-by-two in each round, with the winner moving on to the next round and the loser permanently eliminated from the competition. The tournament proceeds recursively with players advancing into the pool until only one player remains—the tournament winner.

In a single-elimination tournament, the minimal number of rounds is equal to the power to which 2 is raised to get the total number of players. Indeed, if p is the number of players and R the minimal number of rounds, we have:

$$R = \lceil log_2 \, p \rceil$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x.

For example, using the bracket illustrated in Figure 4, there are eight players, hence, the minimal number of rounds will be equal to $\lceil log_2\, p \rceil$ = $\lceil log_2\, 8 \rceil$ = 3.

In a reverse way, the maximal number of players for a particular format of R rounds can be obtained via the formula:

$$p = 2^R$$

The main appeal of the single-elimination tournament is therefore its simplicity. Indeed, this model of organization can be easily constructed and applied when the number of players or teams is large. Moreover, this type of tournament is the fastest way to produce a winner.

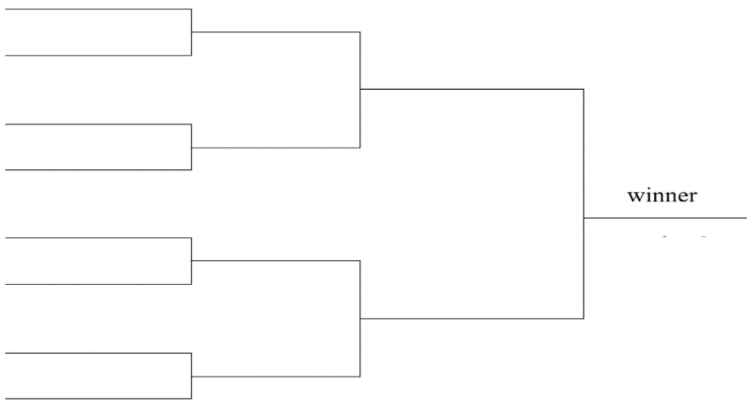

Figure 4: Single elimination bracket with eight participants

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c) In contrast to the single elimination model, in a **double-elimination tournament**, from the second round, a second bracket containing all the losers is introduced, as can be seen at the bottom of Figure 5. Every time a player loses a match, they are put in this second bracket to play at another time.

   Having this bracket allows a player who has lost once to still play in the finals. Indeed, it accepts that one of the best players may have a bad first match or may have been poorly seeded in the single-elimination draw—the double-elimination format ensures that all entries play at least two games (Byl, 2014). However, this model requires more time and organization.

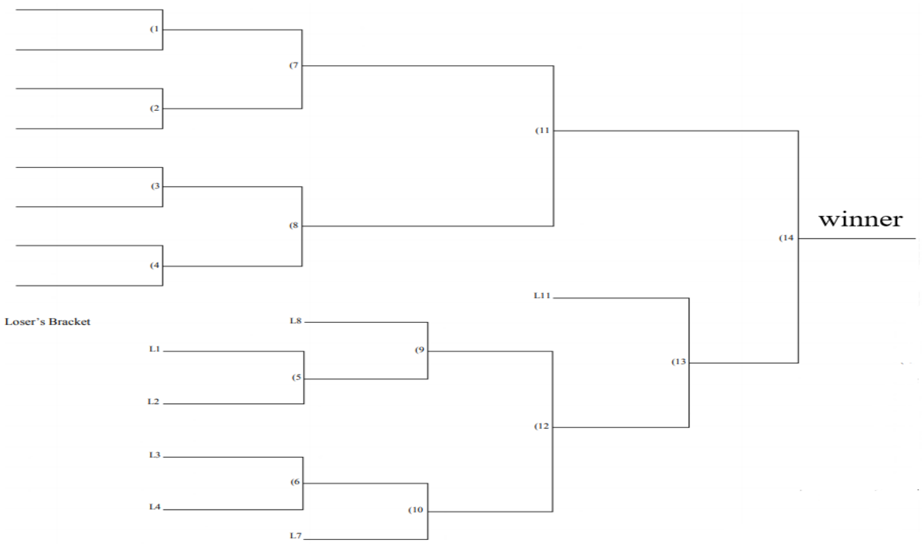

Figure 5: Double elimination bracket with eight participants. Retrieved from https://www.printyourbrackets.com/

1.3.2.1 Which tournament type to choose?

As detailed in subsection 1.3.2, each type of tournament has its own advantages and drawbacks. Each situation, therefore, needs to be analyzed in order to determine which type of tournament is the most suitable. For example, single and double elimination tournaments require fewer games than round-robin tournaments. For a single-elimination tournament with p players, only p-1 games are played, or 2*(p-1) games in the case of a double-elimination tournament, while, in a round-robin tournament, $\frac{1}{2}$ * p * (p-1) games will have to be played. Hence, if games are expensive or time-consuming, or the number of teams is high, a double-elimination or a round-robin tournament may be inappropriate. However, it must also be taken into account that a knockout tournament may, sometimes, result in an unexpected situation, with the strongest player eliminated early in the tournament. These conflicting issues must be balanced prudently in order to ascertain the most suitable type of tournament.

In this thesis, only the case of **random knockout tournaments** is studied. Hereafter, unless otherwise stated, the use of the term "tournaments" refers exclusively to random knockout tournaments.

1.3.3 Tournament structure

In random knockout tournaments, different types of structure, indicating the skeleton of the tournament, need to be distinguished. The structure gives an indication of how the players will be paired, but without stipulating which players will face each other (Edwards, 1991).

As an example, Figure 6 shows two different types of tournament structure which use the same format. As explained in section 1.3.1, the tournament format refers to the number of matches played in each round, whereas the structure refers to the way the matches are placed in the tournament. In Figure 6, both tournaments have a 2-2-1 format but different structures.

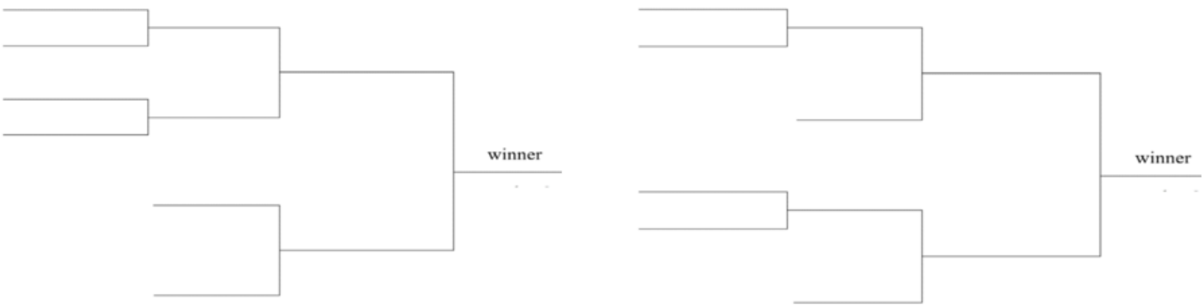

Figure 6: 2-2-1 tournament format with different structures

Within the structures, two major types can be specified: balanced and unbalanced.

a) Balanced structure

In order to define balanced structures, two situations must be distinguished.

i) The number of participants is equal to a power of 2—in this case, the balanced structure was given in Figure 4. The number of rounds, R, is equal to log_2p. The first round counts $\frac{p}{2}$ matches, the second $\frac{p}{4}$ matches, and so on until there is only one game left.

ii) Nevertheless, it often happens that the number of participants is not a power of 2. In this second case, the number of players is equal to $2^r + k$ with $0 \leq k < 2^r$. In this situation, to obtain a balanced structure, k matches are added in the preliminary round, followed by the balanced (i.e. symmetric) structure for the remaining 2^r players. An example of a balanced structure with six players is given in Figure 7.

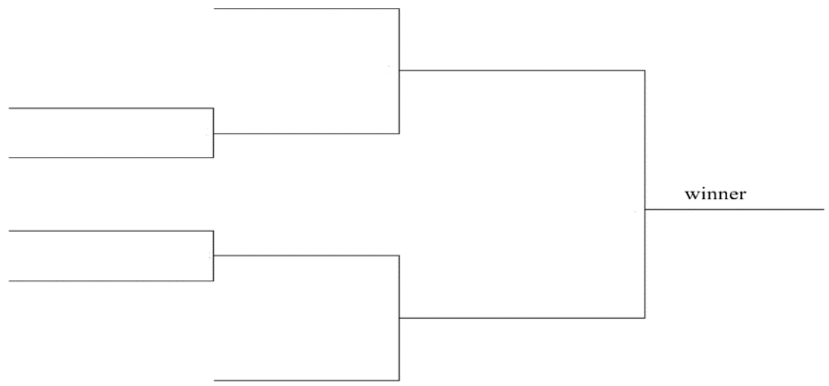

Figure 7: Balanced structure for knockout tournament with six players

b) Unbalanced structure

All other tournament structures that are not balanced are called unbalanced.

There is therefore a multitude of different cases for the unbalanced structure. The extreme case, where there is only one single match per round played, is shown in Figure 8, and is categorized as the totally unbalanced structure.

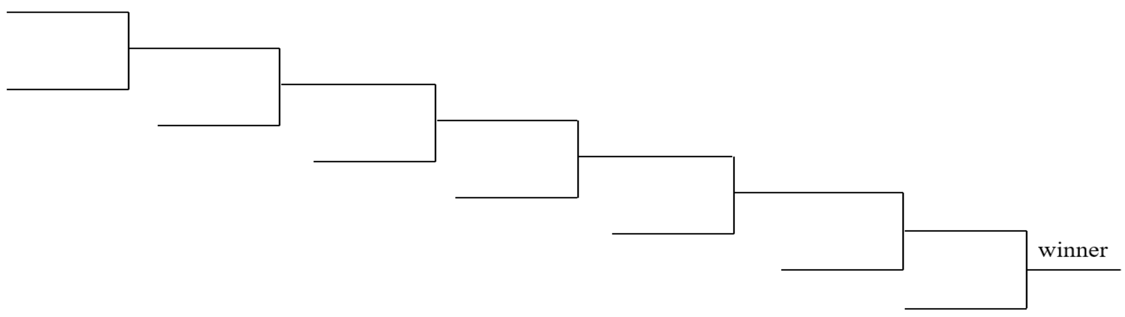

Figure 8: Totally unbalanced structure for knockout tournaments with eight participants

1.4 OVERVIEW

To end the introduction, this section contains an overview to present the structure of this thesis. Chapter 1 presents an outline to the problem, Chapter 2 explains the methodology used and a review of the related literature. The methods used in the study are then described in Chapter 3, along with the obtained results. Finally, Chapter 4 outlines the main conclusions and identifies both limitations to the study and recommendations for further research.

Chapter 2: Literature Review

The literature review is an essential part of the research process. It highlights information, studies, and results previously written on the subject, through scientific articles, books, and articles from specialized journals. It helps to provide a foundation of knowledge on a particular topic and, moreover, to identify links between the current study and related research. Identifying exactly where an intended study sits in the general field of research gives essential perspective to the work (Upstate, University of South Carolina, 2020).

The aim of this thesis is to understand and synthesize some of the fundamental models, applications, and results pertaining to the design of knockout tournaments. To that end, various theoretical positions are presented in section 2.2, along with previous studies.

To begin this overview, section 2.1 briefly presents the methodology used to carry out the literature review.

2.1 METHODOLOGY

To answer the research question, the first task was to analyze and understand the different types of structures used for knockout tournaments.

As mentioned in the introduction, the starting point of this thesis was the article "Random Knockout tournaments", by Adler et al. (2017), which proves, in a particular case, that the balanced format is the one that maximizes the strongest player's chances of victory in a knockout tournament. The author states, "Although we do not have a proof, we conjecture that, among all possible formats, the balanced format maximizes and the one-match-per-round format minimizes the best player's probability of winning the tournament even in the case of general" strengths (Adler et al., 2017, p1). My aim was to support or disprove this conjecture— not by proving it mathematically, like Adler et al., but by using results given by an iterative algorithm.

To gather more relevant information about tournaments and the winning probabilities they involve, the scientific literature on this topic was reviewed, with an emphasis on knockout tournaments. Moreover, an introduction to graph theory and the theory of trees was studied in order to build the different types of tournament structure. More information on this is presented in Chapter 3.

2.2 PRIOR WORK

Going back the oldest literature on the subject, Zermelo (1929) seems to have been the first author to consider paired comparison, a process of comparing individuals in pairs to decide which of each is preferred. Zermelo developed a method based on the maximum likelihood principle, a method of estimating the parameters of a model that selects the set of values maximizing the likelihood function. The author considered sports events, mainly chess, and was one of the first researchers to introduce the ranking theory, restudied thereafter by Wei (1952) in his doctoral thesis.

The Zermelo method was similarly rediscovered by Bradley and Terry (1952), which led them to develop a probability model that allows the computation of the result of a paired comparison. The Bradley-Terry model defines the probability that object j is preferred over object i in the form of pairwise judgments, using this equation:

$$\boldsymbol{V(i > j) = \frac{v_i}{v_i + v_j}}$$

where v_i and v_j are positive real-value scores assigned to i and j.

The Bradley and Terry model is the most commonly used model in sports operational problems, as it also allows researchers to compute the probability that player i will beat player j.

Kendall (1955) applied this same model in the case of sports tournaments by assigning each player a strength. Here, the term "*strength*" is used as a numerical measure to show how good—for example, how strong—a team or a player is. After each stage, he computed the score of each player by giving them the score of every opponent they had beaten and half the score of whoever they were beaten by. Consequently, after each stage, the player ranking was updated.

The notion of ranking is an important part of sports research, as it allows an objective indication of the strength of an individual to be given based on their previous performance (Lasek et al., 2013).

One of the most notorious ranking systems was described by Elo (1977) and is referred to as the "Elo rating system". The basis of this system is that, in the beginning, each player receives a rating, representing a total of points indicating their skills. After each match, the points are updated depending on both the outcome of the match and the opponent's strength. As players with a high rating are expected to defeat weaker players, victory will bring them only a few points. On the other hand, if underdogs win, they take many points. The idea behind the Elo ranking is, therefore, to correct teams' rating points during the tournament's progress (Lasek et al., 2013).

Although the Elo rating was initially created to rank players in chess, it is now regularly used in other domains—for example in the go games, some online games, and even for ranking football teams.

In the context of this thesis, unlike Kendall and Elo, it is assumed that a player's strength is fixed and known. This specification will be essential in Chapter 4, when computing the probability of victory of each player for each match, as the strength is a part of the calculation.

Hence, a player's strength is a key notion for sport tournaments optimization. It helps to determine which player will be considered the best. From this basis, many studies have been performed on the sports tournament's fairness, investigating how to build a tournament so that the highest-ranked player—that is, the strongest—has the highest probability of winning the tournament and the second highest-ranked player has the second-highest probability of winning (Bengston, 2010). This probability that the best competitor wins the tournament, related to the tournament seeding, has been studied by many authors, including David (1959), who built on Kendall's work by taking into account knockout tournaments and by analyzing the winning probability of the top player in a four-player tournament with random seeding.

Glenn (1960) expanded the number of players and also examined the effectiveness of knockout tournaments. He introduced the definition of the "best" player as the one having a probability higher than 1/2 of beating each of the others, an approach that was subsequently used and studied by Searls (1963).

Hwang (1982) presented a new method of seeding. In contrast to the classic approach, this method reseeds players after each round and therefore takes into account their updated strengths. This method will thus tend to benefit the strongest players by favoring them.

Horen and Riezman (1985) also used the notion of "best", by considering the best possible draw under four criteria. They wondered which of the following the best draw would achieve: maximizing the best team's probability of winning the tournament; giving the best team the higher probability of winning; maximizing the probability that the strongest team will only face the second-strongest team in the final round; or maximizing the expected value of the winning team. They finally came to the conclusion that the balanced structure, for a four-player tournament, meets all four criteria and asserted that the strongest player's probability is maximized under the seeding [(1,4)-(2-3)], where the numbering represents the players' ranking.

Appleton (1995) looked at the different types of tournaments to ascertain which ones were optimum. He concluded that the double round-robin tournaments most often result in the strongest team winning the tournament. However, he highlighted the benefits of knockout tournaments by saying that they produce a winner in fewer matches and also involve some excitement in the draw, as they sometimes permit a lower-ranked player to reach the final.

McGarry and Schutz (1997) compared the efficiency of different tournament types such as round-robin and knockout tournaments in their paper "Efficacy of traditional sport tournament structures". Note the difference in the notion of structure. Indeed, they used the term "structure" to refer to the type of tournament, as opposed to the use in this paper, to designate the shape of the tournament bracket. They concluded that "the KO[1] structure is probably the most suitable tournament structure in most cases, given its ranking ability of all players, its promotion of the strongest players and the relatively few games required" (McGarry & Schutz, 1997, p. 74). However, they specified that it is only the case with a fair seeding, asserting that random knockout performs much worse.

Schwenk (2000) focused on finding the fairest seeding for a knockout tournament. He used the term "fair" to explain that "each team's probability of winning should somehow reflect its inherent strength" (Schwenk, 2000, p. 142) and introduced the "cohort randomized seeding". For an eight-player tournament (see Figure 9), the bracket is divided into three cohorts, with the first two players (P1 and P2) placed in the first cohort; C1 = {P1, P2}. The second cohort is C2 = {P3, P4}, and the third cohort is C3 = {P5, P6, P7, P8}. Once all the cohorts are obtained, the players are randomly assigned to a leaf according to their cohorts. For example, P1 could be assigned either to leaf 1 or leaf 5. Schwenk defined this type of seeding as the fairest, as it

[1] Knockout tournament

respects some rules, including the fact that a higher seeded, i.e. weaker, player should not have a harder draw than a stronger player. Hence, Schwenk's method, unlike Hwang's, tends to minimize the best players' favoritism, by minimizing the disproportionate advantage of the best players.

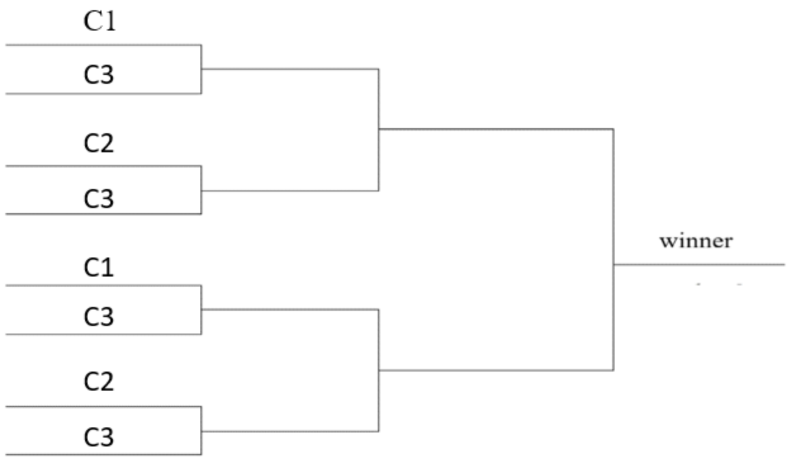

Figure 9: Cohort randomized seeding for eight players

In the references already cited in this section, some researchers have based their results on experiments carried out using algorithms. With advances in computational science, some authors have also focused on the related complexity and on ways of reducing it.

Vu et al. (2009) analyzed the "schedule control" of a tournament—that is, which type of design maximizes the chance of victory for a given player—and the associated computational complexity. Their paper, "On the complexity of schedule control problems for knockout tournaments", has had a significant influence on probability studies in the sports sector, especially those dealing with the subject of sports bribery, as their research included manipulating the tournament seeding so that it benefits a certain player. Moreover, the authors showed that when the number of players is a power of two and the tournament structure is balanced, the problem is NP-hard.

Aziz et al. (2018) in their paper on balanced knockout and double elimination tournaments also analyzed rigged tournaments, showing that verifying whether there is a draw in which a player wins is a NP-complete problem.

As can be noticed in the above-mentioned articles, most studies have analyzed seeded tournaments, few of them deal with random tournaments. The main reason is that in a totally random tournament, it may happen that the strongest player meets the second strongest player in the first match. This would imply that in a knockout tournament a very strong player could be eliminated in the first round. By the same logic, a weak player, or even the weakest player, would have an increased probability of reaching the final. This would lead to uninteresting matches or finals. Moreover, in the sports field, spectator entertainment is paramount and valuable from a financial point of view. A first-round match involving the two best players or a final in which a good player faced one of the weakest ones would therefore be uninteresting and financially disastrous. This is why, in most cases, tournament organizers have a clear preference for seeded tournaments (Schwenk, 2000).

Nevertheless, some authors have still studied random tournaments. Edwards (1991), for example, conducted a detailed study on knockout tournaments. His paper "Random knockout tournament" was widely used during the writing of this thesis and is, therefore, quoted several times throughout.

Marchand (2002) compared the probability that a strong player wins the tournament with a random seeding versus a standard seeding. He came to the conclusion that, contrary to expectations, in terms of probabilities, the outcome of a random tournament is close to that of a seeded tournament.

Ross and Ghamami (2008) also considered random knockout tournaments by taking into consideration the problem of using simulations to estimate the win probabilities of each player. They used various estimators to compare the different draws and concluded that the method of "observed survivals" is the most efficient one. They used the term "observed survivals" to denote the set of players still alive after each round r. Those survivals are the states of a Markov chain—a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event, according to the Oxford dictionary ("Markov chain", n.d.). Using the Markov model gives the authors a considerable advantage of speed and accuracy when producing results.

Karpov (2016) performed a study to determine which type of seeding was optimal for the strongest players in case of knockout tournament with a number of players equal to a power of two. He introduced the notion of the "equal gap seeding", i.e. a "unique seeding that, under the linear domain assumption, maximizes the probability that the strongest participant is the winner, the strongest two participants are the finalists, the strongest four participants are the quarterfinalists, etc." (Karpov, 2016, p. 706)

In all the aforementioned articles, only classic type tournaments were studied. It was Maurer (1975) who first started to consider different formats.

The term "*classic*" is used here to describe a tournament that has p players and a balanced structure. As a reminder, the balanced structure can be defined as a structure, including p players with $p = 2^R + \mathrm{k}$ where R is the total number of rounds and k additional players, for which there are k matches in the preliminary round, followed by the balanced (i.e., symmetric) structure for the remaining 2^R players. The $p - (2 * k)$ players that did not play during the first round have what is called "bye", as illustrated in Figure 10.

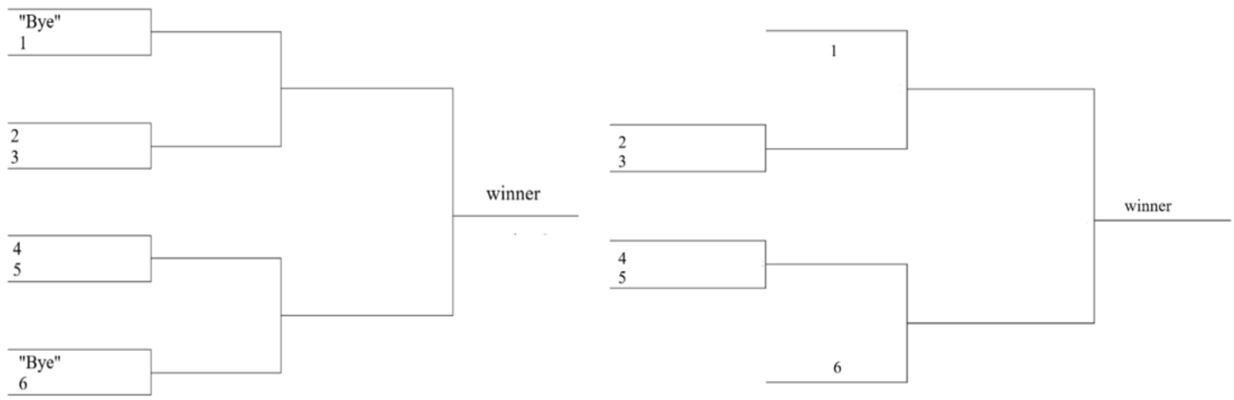

Figure 10: Six-player tournament structure with two "byes"

A multitude of other structures can be designed in order to build a tournament for p players, some being more eccentric than others. The generation of all these different structures is explained in Chapter 3.

As previously stated, the Adler et al. (2017) paper on random knockout tournaments is of paramount interest to this study. Hence, a large part of this section is dedicated to an analysis of this article. The authors proved that, in a random knockout tournament, in the special case where $v_1 > v_2 = v_3 = \ldots = v_p$, it is the balanced format that maximizes the strongest player's chance of winning. This case was first studied by Maurer (1975), who proved, but in a different way, that for this particular case, the balanced structure is, indeed, optimal.

Adler et al. (2017) also demonstrated, by analogy, that totally unbalanced structure—that is, the one match per round format—minimizes the strongest player's chances of winning. Moreover, the authors gave both an upper and a lower bound for each participant, the upper bound giving the probability that the weakest player will win the tournament, and the lower bound giving the probability that the strongest player will win.

Another well-used article was produced by Bengtson (2010), who gave more explanation on seeding and its impact on the odds of winning a knockout tournament. Like Adler et al. (2017), Bengtson used the Bradley-Terry model that returns the probability of participant i beating participant j, noted as V_{ij} in this equation

$$V_{ij} = \frac{strength(i)}{strength(i) + strength\,(j)}$$

From that rule, the preference matrix P, a matrix containing the probability that a player beats another, can be derived. Applied to an eight-player tournament, this is denoted thus:

$$\mathrm{P} = \begin{bmatrix} V11 & \cdots & V18 \\ \vdots & \ddots & \vdots \\ V81 & \cdots & V88 \end{bmatrix}$$

David (1959) showed that this preference matrix P follows the strong stochastic transitivity. This results in ∀i j k,

$$if\, V_{ij} \geq 0.5 \; and\, V_{jk} \geq 0.5 \; then\; V_{ik} \geq 0.5$$

This means that if the probability of participant i beating participant j is greater than or equal to 0.5, and the probability of participant j beating participant k also greater than or equal to 0.5, the probability of participant i beating participant k, by the principle of transitivity, will similarly be greater than or equal to 0.5.

This paper only considers the strong stochastic transitivity matrix, and presumes that the players are ranked in the order $p_1, p_2, \ldots, p_n$ from the strongest to the weakest (Hwang, 1982). Consequently, as the matrix P follows the strong stochastic transitivity, players can be ranked according to their strengths, which will be useful later when calculating each player's probability of winning. Player p_i will be ranked above player p_j, indicating that player i is stronger than player j if p_i has a better chance of winning against p_j.Moreover, Bengtson used the probability that i win the round r as

$$W_{ir} = W_{i,r-1} \left[\sum_{k=v}^{u} P_{ik} W_{k,r-1}\right] where\; W_{i,0} = 1 \; and\; r > 0$$

with

$$v = S(i,r) = 1 + 2^{r-1} + 2^{r-1} \left\lfloor \frac{i-1}{2^r} \right\rfloor - 2^{r-1} \left\lfloor \frac{i-1}{2^{r-1}} \right\rfloor$$

$$\text{and } u = v + 2^{r-1} - 1$$

In the above formula, u and v are used to represent the upper and lower limit of the possible opponents of i. These limits indicate all possible opponents of a player i in a round r.

For example, in an eight-player tournament with three rounds (R=3), as shown in Figure 10, during the first round, the player located in the first leaf can only meet the player of the second leaf. In the second round, they could meet either the player located on the 3rd or the 4th leaf. Finally, in the third round, they could meet either the player located on the 5th, 6th, 7th, or 8th leaf.

Thus, for the player located in leaf 1:

- S(1,1) = 2 as v = 2 and u = 2
- S(1,2) = 3,4 as v = 3 and u = 4
- S(1,3) = 5,6,7,8 as v = 5 and u = 8

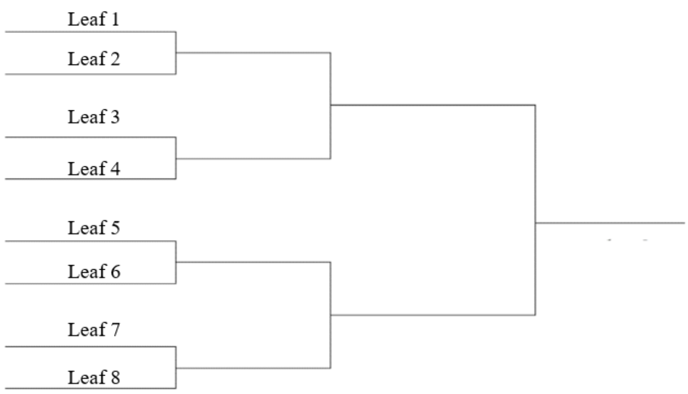

Figure 11: Tournament bracket with eight leaves

Using the same example, W_{11} is the probability that player 1 wins round 1 or the probability that player 1 beat player 2. W_{12} is the probability that player 1 wins round 2 or the probability that player 1 wins round 1 and beats player 3 or player 4.

These formulas, first introduced by Edwards (1991) in his doctoral thesis about knockout tournaments, allow the computation of the probabilities that a player has to win each match. By taking the probability of winning the last match, i.e. the match in round R, the probability of winning the tournament is obtained.

Chapter 3: Theoretical Framework and Development

3.1 BINARY TREES

3.1.1 Preliminary theory

Before investigating the thesis topic further, definitions and additional explanations of the graph theory and the theory of trees are required. This study uses perfect binary trees with n external nodes as a representation of a p players tournament structure.

i) Graph Theory

The theory presented in this section is taken from the lecture notes on "Graph theory" by T. Harju from the Department of Mathematics at the University of Turku (Harju, 2012).

Let V be a finite set, and denote by

$$E(V) = \{\{u, v\} \mid u, v \in V, u = v\}$$

Definition 2. *A pair G = (V, E) with E $\subseteq$ E(V) is called a* ***graph*** *(on V). The elements of V are the vertices of G, and the elements of E are the edges of G. The vertex set of a graph G is denoted by VG and its edge set is denoted by EG. Therefore G = (VG, EG).*

ii) Theory of Trees

Unless specifically mentioned, the theory presented in this section is taken from the lecture notes on "Advanced computation" by P. Geurts from the Montefiore Institute at the University of Liège (Geurts, 2018).

Definition 3. *A **tree**[2] T(N,E) is a connected acyclic graph where*

- *N is a set of nodes*
- *$E \subset N \times N$ is a set of edges*

with the following properties:

- T is connected and acyclic, i.e., all the nodes are connected and there is no cycle within the tree.
- If T is not empty, there exists a distinguished node called the root node. This root is unique.
- For each edge (n_1, n_2) in E, if n_1 is on the unique path from the root to n_2, then n_1 is called the parent of n_2, and n_2 is called a child of n_1.
    - The root node of T does not have a parent
    - The other nodes of T have only one parent node
- The path is a sequence of nodes $n_1, n_2, \ldots, n_N$ such as for all i $\in [1, N-1]$, (n_i, n_{i+1}) is an edge of the tree.

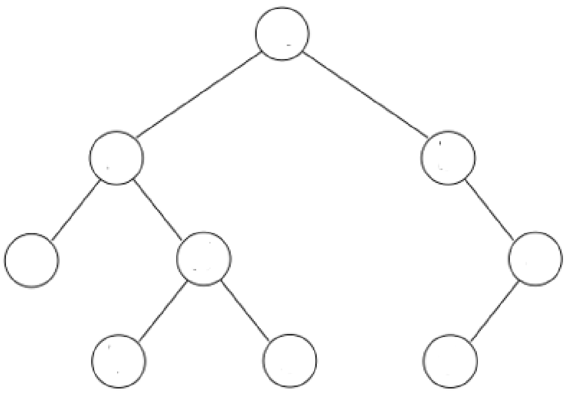

Figure 12: A rooted tree with nine nodes

There is also some additional and specific terminology related to theory of trees:

- If n_1 is the **parent node** of n_2, n_2 is the child of n_1.
- Two nodes n_1 *and* n_2 having the same parent node will be **siblings**.
- A node having at least one child will be considered as an **internal node**.
- An **external node**, i.e., a non-internal node, is a leaf.

[2] In graph theory, a rooted tree

- The **height** of a node is the number of edges of the longer path from this node to a leaf. The height of the tree is the height of its root.
- The **depth** of a node is the number of edges on the path to reach the tree root.

Definition 4. *A **binary tree** is a tree having the following property:*

- *Each of its nodes has no more than two children.*

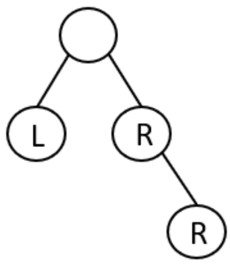

Figure 13: A binary tree with four nodes

From Figure 13, the following point can be added:

- Each child is either a left or a right child.

***Definition 5**. A **full binary tree** is a binary tree in which every external node has exactly two children.*

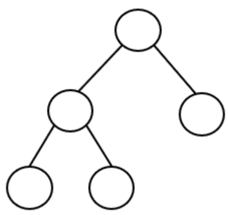

Figure 14: A binary tree with two internal nodes and three external nodes

__Definition 6__. A __perfect binary tree__ is a full binary tree in which all the leaves have the same depth.

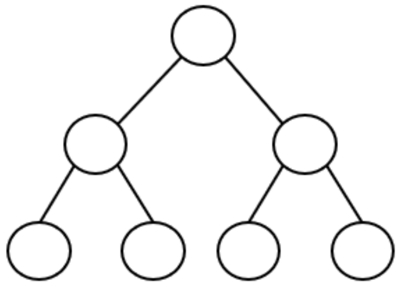

Figure 15: A perfect binary tree with three internal nodes and four external nodes

A **full binary tree** has these properties:

- The number of internal nodes, n, is equal to $\frac{N-1}{2}$, where N is the total number of nodes.
- The number of external nodes is equal to the number of internal nodes plus 1.
- The number of nodes at the depth i is smaller than or equal to 2^i.
- The height h of the tree is smaller than or equal to the number of internal nodes.

Figure 16 shows the distinction between a full binary tree and a complete binary tree, both with four external nodes. These concepts are essential to the next sections, which deals with different tournament structures.

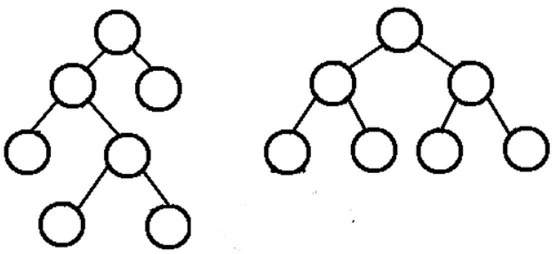

Figure 16: Full binary tree vs complete binary tree with four external nodes

Definition 7. *A **labeled tree** is a tree in which each node is assigned a unique number from 1 to |N| (Weisstein, Labeled Tree, n.d.).*

In order to distinguish two labeled trees, the way in which the nodes are labeled is key. The mathematician Cayley (1889) was one of the first to discuss the mathematical theory of trees, and, in his paper "A theorem on trees", he described the number of different rooted labeled trees. He concluded that the number of trees on n labeled nodes is n^{n-2}. Indeed, as can be seen in Figure 17, those three trees, although similar in structure, are different in terms of labeling.

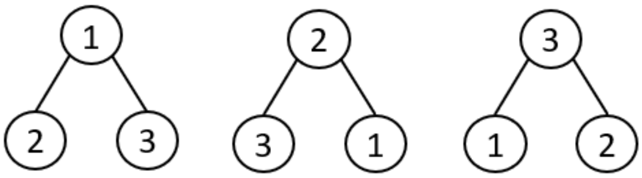

Figure 17: Labeled tree with three nodes

Definition 8. *A **subtree** is a tree which is a child of a node.*

In Figure 18, the section circled in red, B, represents the subtree of the binary tree A.

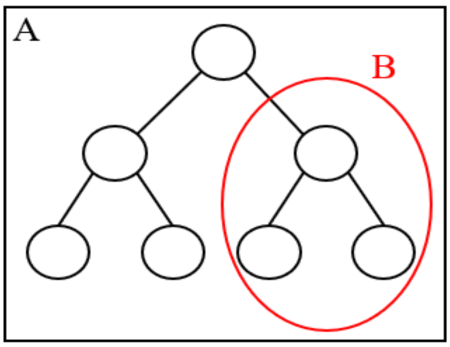

Figure 18: A binary tree and its subtree

Definition 9. *An **ordered tree** is a tree in which the order of the subtrees is significant (Weisstein, Ordered Tree, n.d.).*

More precisely, a full binary tree is ordered if the children of each internal node are labeled left and right, respectively. Hence, two ordered trees are considered as distinct if the children of at least one node are labeled differently.

Several authors have been investigating the number of different full ordered binary trees and, among them, Stanley (1997), in his book dealing with combinatorial problems, described the relationship between the Catalan number and many contexts, including:

- The number of different ways to correctly match n pairs of parentheses:

  ((())) ()(()) ()()() (())() (()())

- The different ways in which a product of n different ordered factors can be calculated by pairs:

  ((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd))

- The total number of full ordered binary trees.

This study will only focus on the latest examples. Indeed, the number of full binary ordered trees with n internal nodes is given by the n^th Catalan number and can be computed with this formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

For example, for n = 3, $C_3 = 5$, which means that five distinct full ordered binary trees with three internal nodes can be identified (see Figure 19).

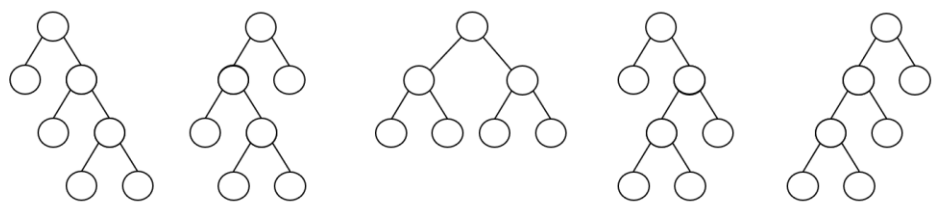

Figure 19: All possible full ordered binary tree with three internal nodes

Using the previously defined terms, it can be stated that the two left trees and the two right trees have the same format, 1-1-1. Moreover, those four trees are isomorphic.

Definition 10. *Two graphs (N,E) and (N',E') are* **isomorphic** *if |N|=|N'| and if the nodes of N and N' can be relabeled with the elements of {1,2,...,|N|} in such a way that the resulting labeled graphs become identical (same set of nodes, same set of edges).*

That is, (N,E) and (N',E') are isomorphic if they are identical apart from the label of their nodes. In terms of sports tournaments, it is irrelevant whether a leaf of the tree is pointing to the right or to the left. This is explained in more depth in section 3.1.2.

The number of non-isomorphic trees with three internal nodes can be reduced to two (see Figure 20), and these can be termed **unordered binary trees**.

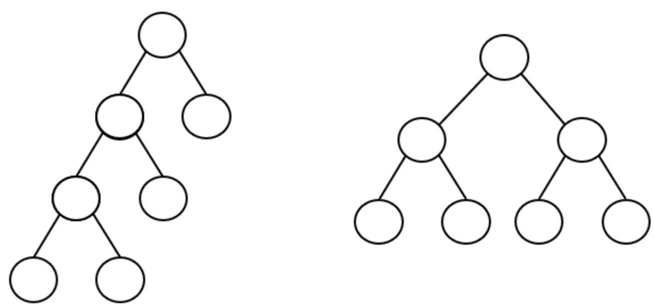

Figure 20: Unlabeled tree with three internal nodes

"Unordered" simply means that the order is insignificant. As can be seen in Figure 21, both trees are equivalent if they are considered as unordered trees. Indeed, they differ only in the respective ordering of their subtrees. However, these trees would have been seen as distinct if they were considered as ordered trees.

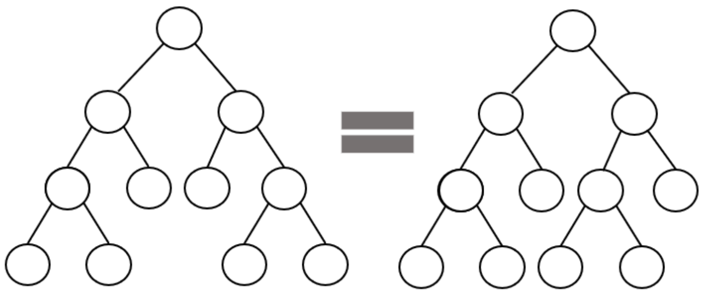

Figure 21: Equivalence in unordered trees

3.1.2 Tournaments as Binary Trees

The relationship between a knockout tournament and graph theory is easily noticeable, as can be seen in Figure 22, a 90-degree rotation to the right of a binary tree will result in a tournament representation. Each of the p players is first placed on one on the n+1 external nodes. Players with the same father node—siblings—compete against each other; the losing player is eliminated and thus removed from the tree, while the winning player advances to the top. The winner will be the player who succeeds in climbing up to the root of the tree.

A tournament structure can, therefore, be built using a full binary tree, but with some small modifications when the number of players is not a power of two. This is explained more in detail in the section 1.6.1.

Binary tree knowledge has thus allowed several authors to manipulate the structures and to obtain different probabilities in the context of sports tournaments. Therefore, for the sake of simplicity, in this paper, the term "binary trees" means "full binary trees".

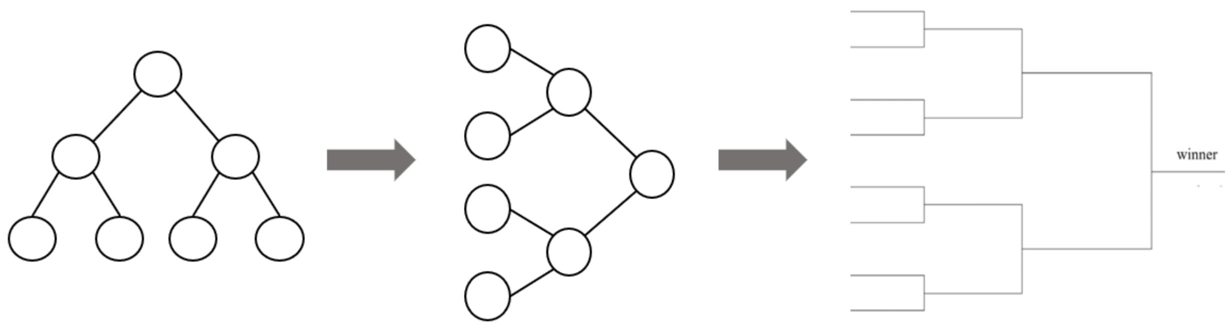

Figure 22: From a binary tree to a tournament bracket

As previously stated, Maurer (1975) was the first to analyze non-classical tournaments by considering different structures. Indeed, he proved that, for a random structured format, in the particular case where $v_1 > v_2 = v_3 = \ldots = v_p$, meaning that one player has a higher strength while all the others have the same strength, it is the balanced format that maximizes the stronger player's probability of winning the tournament. To prove it, he used the fact that a random format for p players is a combination of a random format of fewer players. This proposal is used in section 3.2.2, in the design of the algorithm used in this study, using a recursive function.

Moreover, in his paper, Maurer (1975) counted the number of different tournament structures for p players using this formula:

$$A_p = \sum_{i=1}^{p-1} A_i * (A_{p-i} + \delta_{i,p-i})$$

$$with\ A_1 = 1$$

where $\delta_{i,j}$ denotes the Kronecker's delta, a two-variables function that returns 1 if i=j and 0 otherwise.

Table 1 illustrates how this works for a tournament with up to 12 players.

p (number of players)	A_p
2	1
3	1
4	2
5	3
6	6
7	11
8	23
9	46
10	98
11	207
12	451

Table 1: Number of tournament structures with up to 12 players

Furthermore, Maurer (1975) added an additional formula that, within all the possible structures, allows the balanced ones— B_p—to be obtained:

$$B_p = \sum_{i=l}^{u} B_i * (B_{p-i} + \delta_{i,p-i})$$

$$with\ B_1 = 1$$

where for

p = $2^R + k$ with 0 $\leq k < 2^R$:

- l = max (2^{R-1}, k)
- u = min ($2^{R-1} + k, 2^R$)

The outcome is shown in Table 2.

p	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
$\boldsymbol{B_p}$	1	1	1	1	1	2	1	1	1	3	3	5

Table 2: The numbers of balanced structures for tournaments involving up to 12 players

In a similar manner, Chung and Hwang (1978) represented a knockout tournament with p players as a full binary tree with n+1 external nodes. They conjectured that, in the case of any knockout tournament with an organized structure where all the participants are equally likely to be placed in the bracket—i.e., a random knockout tournament—the stronger player has a higher probability of winning the competition. Indeed, the authors assumed that if $V_{ij} > \frac{1}{2}$, then the probability that i wins the tournament will be higher than or equal to the probability that j wins the tournament.

However, this was later contradicted by Israel (1981), who proved that for certain types of structures, the strongest player does not necessarily have the highest likelihood of success. The author showed, by a counterexample with a random single elimination of 17 players, that a player with a lower strength may have a higher winning probability than their stronger opponents.

Later, in his doctoral thesis "The combinatorial theory of single-elimination tournaments", Edwards (1991) took up many of the principles previously cited in order to make a complete and thorough study of knockout tournaments. Like Maurer, he associated the notion of a knockout tournament with an unordered binary tree. Actually, as can be seen in Figure 19, the two trees on the left are isomorphic to the two trees on the right. This distinction allows a significant reduction in the number of different trees.

In the same way that the number of ordered binary trees with n internal nodes[3] is represented by the nth Catalan number, Etherington and Wedderburn found the total number of unordered binary trees[4] with their so-called sequences of Wedderburn–Etherington numbers. This sequence can be obtained with:

$$U_{2n-1} = \sum_{i=1}^{n-1} U_i \, U_{2n-i-1} \quad \textit{for even numbers}$$

$$U_{2n} = \frac{U_n\,(U_n+1)}{2} + \sum_{i=1}^{n-1} U_i \, U_{2n-i} \; \textit{for odd numbers}$$

$$\textit{with } U_1 = 1$$

Table 3 shows the difference between the number of ordered (C_n) and unordered (U_n) binary trees with n internal nodes.

n (=internal nodes)	C_n	U_n
2	2	1
3	5	2
4	14	3
5	42	6
6	132	11
7	429	23
8	1430	46
9	4832	86
10	16796	207
11	58786	451
12	208012	983

Table 3: The numbers of ordered and unordered binary trees up to 12 internal nodes

[3] OEIS: A000108
[4] OEIS: A001190

It can be seen that the right column values were previously obtained by Maurer, who defined them as the number of different possible structures of a knockout tournament with p players. As a reminder, p = n+1 with n the number of internal nodes if the tournament is seen in terms of a binary tree.

The main advantage of obtaining unordered trees is therefore to reduce the number of structures possibilities.

3.2 TOURNAMENT STRUCTURE

3.2.1 Unlabeled tree as a tournament structure

One objective of this study is to be able to generate all possible tournament structures with p players—that is, all full unordered binary trees with n+1 external nodes. To do so, full and perfect binary trees with 2^R leaves are used, and players planted into trees' leaves.

When the number of players is a power of two, it is only necessary to place these p players on the n external nodes—the leaves—of the tree. When the number of players is not a power of two, the dummy player technique is used.

This technique, initially used by Searls (1963) and later by Chung and Hwang (1978), Hwang (1982), and Edwards (1991), consists of adding dummy players—players with zero strengths—in order to transform a full binary tree into a perfect binary tree. The number of dummy players, D, is computed by $2^R - p$, with p being the total number of players and R the total number of rounds to be played and obtained by

$$R = \lceil log_2 p \rceil$$

$$D = 2^R - p$$

For example, to build a tournament with six players, the following applies: $R = \lceil log_2 p \rceil = \lceil log_2 6 \rceil = 3$. It is therefore necessary to add $2^R - p = 2^3 - 6 = 2$ dummy players. Those dummy players can be placed in several places in the tournament bracket. Figure 23 shows a possible example where the dummy players are represented in red.

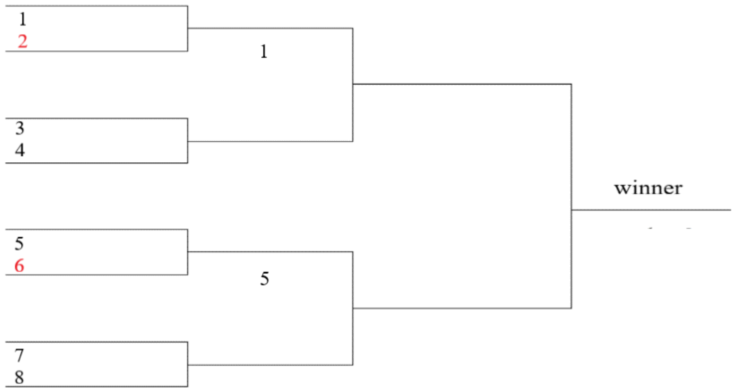

Figure 23: Example of a six-players structure with two dummy players

The outcome is therefore similar to that shown in Figure 9, where two "byes" were inserted. Indeed, inserting the two dummy players, with their zero strengths, will ensure that their opponent wins the match. Using them also allows the use of a perfect binary tree structure, which makes the calculations easier.

However, inserting players with a zero strength will have an impact on the preference matrix P. Indeed, the probability that player i beats player j is obtained by

$$V_{ij} = \frac{strength(i)}{strength(i) + strength\,(j)}$$

For computing the probability that a dummy player beats another dummy player, as division by 0 is not allowed, one will have to win with a probability of 1, and thus 1s and 0s have to be added in order to obtain a correct preference matrix.

For instance, a tournament with three players, two real ones and one dummy, looks like this:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.5 & P_{12} & 1 \\ P_{21} & 0.5 & 1 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Hence, the question that arises is where to place these dummy players in order to obtain all possible unlabeled binary trees and thus all possible structures for a tournament of p players.

3.2.1 Generation of Tournament Structure

The generation of ordered binary trees has been explored in the literature many times. For example, Pallo (1986) introduced weight sequences—sequences of positive integers that characterize the binary tree—and used them in order to lexicographically generate binary trees. Lucas et al. (1993) presented a recursive algorithm which aimed to generate different binary trees by a single edge rotation.

The literature on the generation of unordered binary trees is more limited; but Furnas (1984) presented several methods for randomly generating different types of trees and proposed a possible algorithm. Pallo (1989) introduced the canonical weight coding—a sequence of digits representing the number of leaves in the left (or right) subtree—to obtain a unique representation of a tree T. This work on canonical weight coding was later continued by Effantin (2004). In their papers, Pallo (1989) and Effantin (2004) both presented an iterative algorithm that formulated the next canonical weight coding based on the previous one.

A third method, and the one that is used in this thesis, was suggested by Edwards (1991) who represented each tournament structure by a label. His method consists of giving a label—that is, a unique sequence of digits—representing each of the possible full unordered binary trees with n+1 external nodes. The p players of a tournament will be placed on each of the n+1 external nodes of a perfect binary tree.

This label is composed only of 2s, 1s, and 0s.

- A "2" means that two "real" players, i.e. not dummy, are placed in the bracket
- A "1" means that one real player faces a dummy player
- A "0" means that two dummy players are placed in the bracket

As an example, Figure 24 shows the sequence "2210" for a five-player game with the dummy players in red.

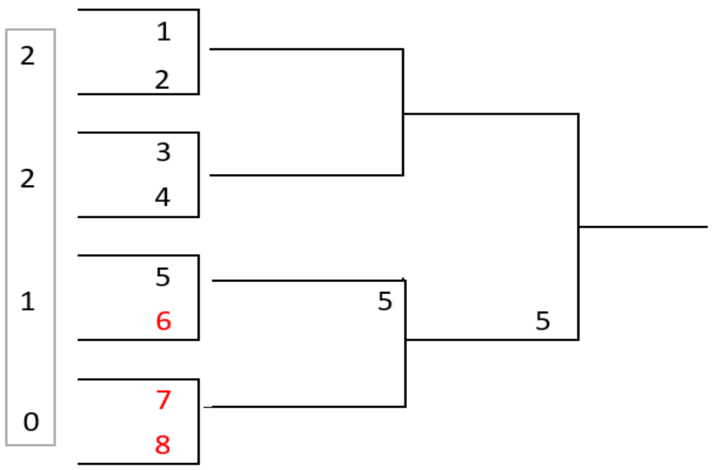

Figure 24: Tournament structure 2210

As each digit of the label corresponds to a match, the length of the label depends on the number of rounds within the tournament structure. Indeed, depending on the structure, there may be between $\lceil log_2\, p \rceil$ and p-1 rounds.

If Rmin is the minimum number of rounds and Rmax is the maximum number of rounds for a tournament with p players, the following applies:

- Rmin = $\lceil log_2\, p \rceil$
- Rmax= p-1

There will therefore be several labels of length 2^x, where $Rmin \ \leq \ x \ \leq \ Rmax$ and x ∈ N.

As an example, for a five-player tournament, Rmin = 2 and Rmax=3, leading to two possible label lengths: 4 and 8 (see Figure 25).

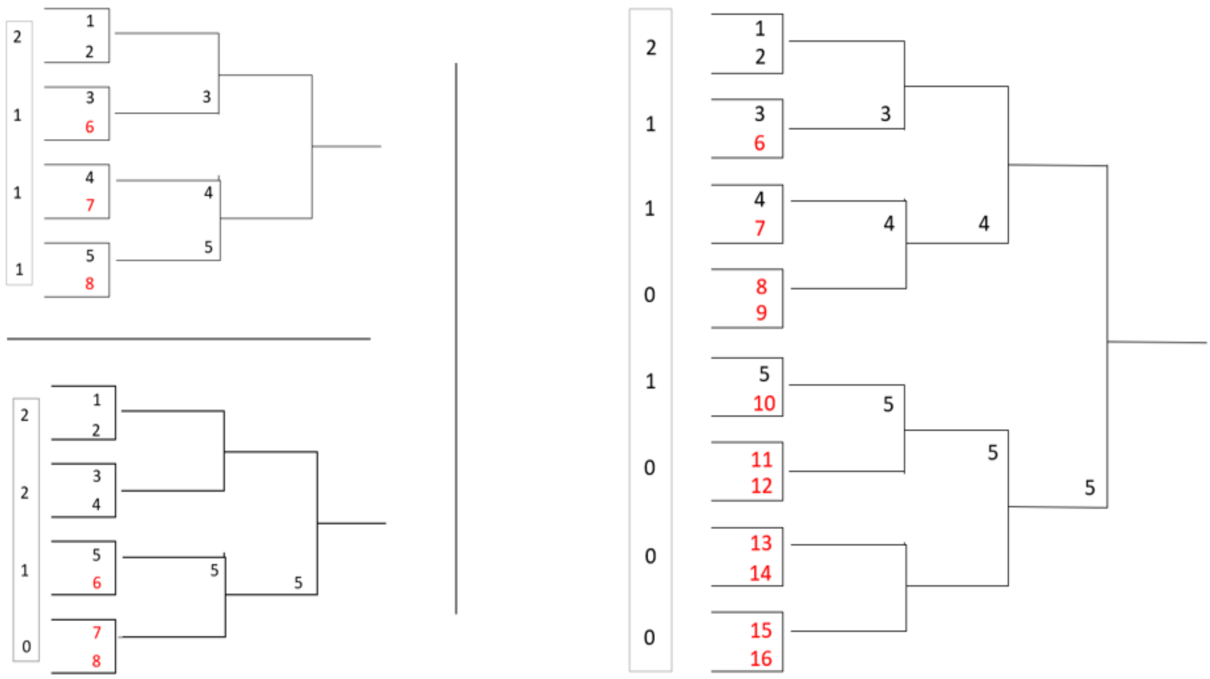

Figure 25: The different labels and related structures for five players

In order to identify these different structures, a set of rules, given by Edwards in Table 4, needs to be respected. In section 3.2.3, based on these rules, an algorithm giving all the possible sequences of tournament structures is presented.

1. A label consists of a string of zeroes, ones, and twos of length 2^{r-1}.
2. The sum of the digits in a label is t.
3. Every label starts with a two.
4. A two cannot be followed by a zero.
5. The sum of the digits in the first half of a label is greater than or equal to the sum of the digits in the second half of a label. This rule also applies to each quarter, eighth, etc., that is, the sum of the digits in the first quarter of a label is greater than or equal to the sum of the digits in the second quarter of a label and the sum of the digits in the third quarter of a label is greater than the sum of the digits in the fourth quarter of a label.
6. If the sum of the digits is the same in both halves of a label, then the sum of the digits in the first quarter is greater than or equal to the sum of the digits in the third quarter. This rule also applies to eighths, sixteenths, etc., that is, if the sum of the digits is the same in all four quarters of a label, then the sum of the digits in the first eighth of a label is greater than or equal to the sum of the digits in the fifth eighth of a label and the sum of the digits in the first sixteenth of a label is greater than the sum of the digits in the ninth sixteenth of a label.

Table 4: Rules for labeling tournament structures with R rounds and t teams. Reprinted from The Combinatorial Theory of Single-Elimination Tournaments, by Christopher Edwards (1991)

3.2.2 Tournament Structure Algorithm

In order to facilitate the understanding of the method used, this subsection outlines the main principles, and details each of them. The main principles are as follows:

- Generate all the full unordered binary trees sequences using Edwards's method
- Randomly assign strengths to players and randomly place them in the bracket
- Compute the probability of each player winning each round
- Deduce which structures minimize or maximize the strongest player's probability of winning

3.2.2.1 Full Unordered binary trees sequences

The method devised by Edwards (1991) is used to obtain all the possible labels representing the full unordered binary trees. However, before obtaining the final tournament structure labels, several steps are necessary.

The first step is to randomly generate labels containing the numbers 2, 1, and 0. Once all the possible labels sequences are obtained, they need to be sorted, based on Edwards' labeling rules, in order to delete the labels that were not allowed.

The outcome returns all possible tournament structures with p players (see Figure 26). The number of different structures for p players follows the Wedderburn–Etherington sequence.

As written in the first point of Table 1, the length of a label is 2^{R-1}, with R the number of rounds. Because unclassical structure is being analyzed, the number of rounds will vary between Rmin and Rmax.

- Rmin = $\lceil log_2\, p \rceil$
- Rmax= p-1

As a label contains only 2s, 1s, and 0s, and as all the possibilities are required, it is necessary to check that, for all R, the number of possible labels is equal to 3^{2^R} with $Rmin \leq R \leq Rmax$.

Hence, the higher the number of rounds, the longer the label. As an example, with six players, the maximal number of rounds is five, and the maximal length of the label is therefore $2^{5-1} = 16$. This produces 3^{16} = 43 054 721 possible labels, which is computationally difficult to generate.

This is why, for a number of rounds greater than two, a recursive function is set up.

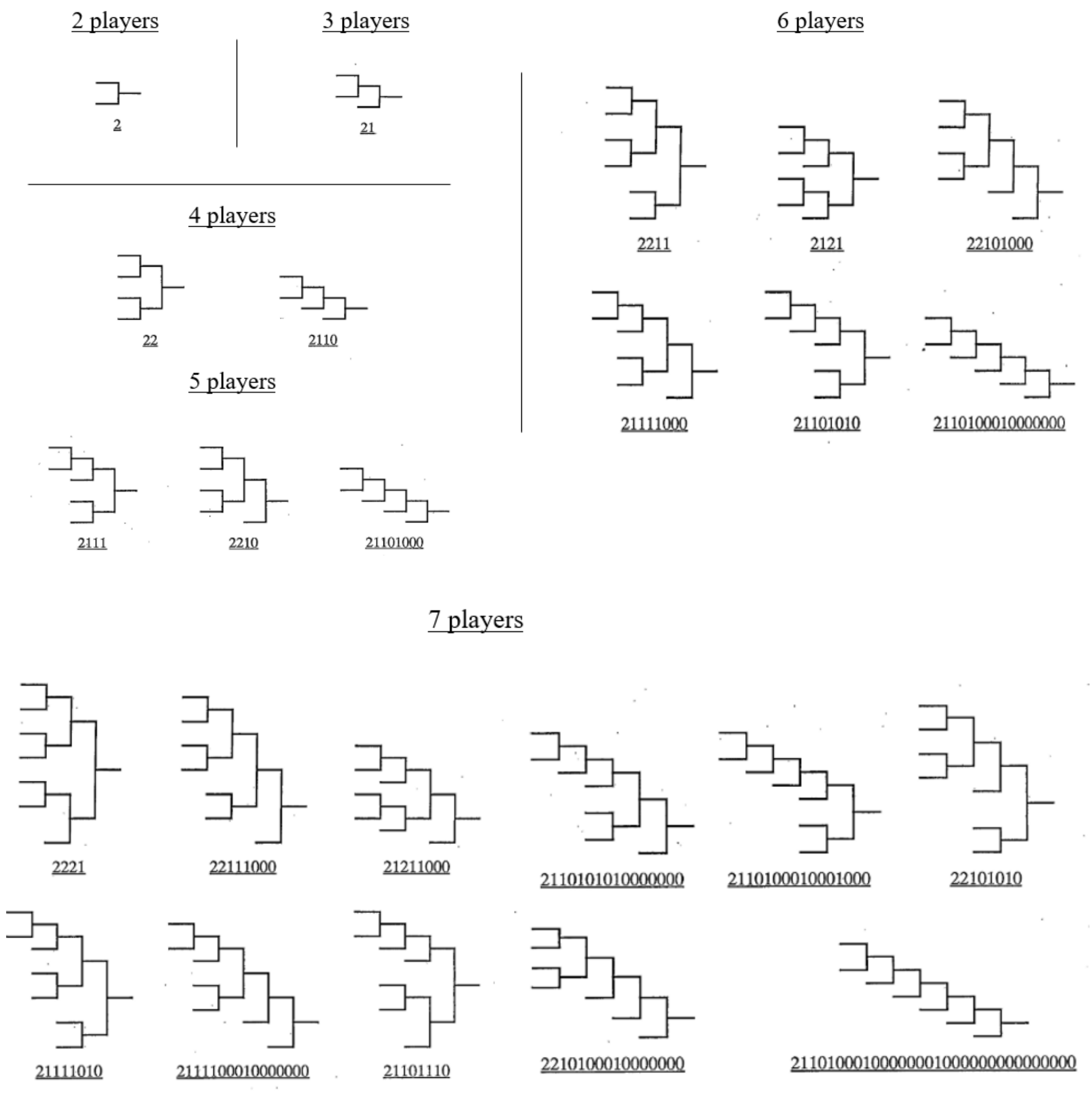

Figure 26: Tournament structure for up to seven players. Reprinted from The Combinatorial Theory of Single-Elimination Tournaments, by Christopher Edwards (1991)

i) Recursive function trick

As previously stated, the algorithm randomly generates sequences of numbers between 0 and 2. Consequently, there is an exponential number of possibilities when the number of rounds, and thus the length of the sequence, increases. To bypass this difficulty, a recursive function was implemented.

An analysis of the structure of a tournament shows that the left half of the label, representing the top table of a bracket, is a substructure of a label with fewer players.

2 players:
- 2

3 players:
- 21

4 players:
- 22
- 2110

5 players:
- 2111
- 2210
- 21101000

6 players:
- 2211
- 2121
- 22101000
- 21111000
- 2110100010000000

7 players:
- 2221
- 22111000
- 21211000
- 21101110
- 22101010
- 2111100010000000
- 2110101010000000
- 2110100010001000
- 2210100010000000
- 21101000100000001000000000000000

Figure 27: Labels for tournaments with up to seven players

As an example, Figure 27 shows the different labels for tournaments involving up to seven players. It can be observed that for the sequences of seven players and five rounds (i.e. length 16), the left part of the label, highlighted in yellow, comes from the eight-length labels of fewer players.

Hence, in the algorithm, from a number of rounds greater than two, the recursive function "use_previous_left" is used to generate the left half of the label.

The rules used in the function use_previous_left(p,r), with p the number of players and r the number of rounds with $3 < r$ are as follows: for any p, store, in a list, the final authorized labels of r-1.

In the same way, to generate the right part of the label, the function "use_previous_right" is used.

The rules used in the function use_previous_right(p,r), with p the number of players and r the number of rounds with $3 < r$ are as follows:

1) For any p, store in a list the right half of the final authorized labels of r-1 and add one label full of 0s.
2) Concatenate the right halves together.
3) Perform skimming, based on Edwards's rules.

The function "use_previous_full" will concatenate the left and the rights part to obtain full labels and, after the skimming with Edwards' rule, returns the accepted labels for p players and r rounds.

Taking the example of Figure 27, the situation for seven players and five rounds is as follows:

- use_previous_left(7,5) returns:

1) The accepted full labels for four rounds
    - [21101000]
    - [21101010]
    - [21111000]
    - [22101000]

- use_previous_right(7,5) returns:

1) The accepted right halves labels for four rounds and the label full of 0s
    - [1000]
    - [1010]
    - [1110]
    - [0000]
2) After the concatenation between each one, and skimming by Edwards' rules
    - [10001000]

- [10101000]
- [10000000]

- use_previous_full(7,5)
  - ✓ [2110100010001000]
  - ✓ [2110101010000000]
  - ✓ [2111100010000000]
  - ✓ [2210100010000000]

Using the two recursive functions, use_previous_left(p,r) and use_previous_right(p,r), enables larger length labels to be generated. Indeed, without it, the algorithm struggles in generating labels with a length greater than 8. With those improvements, labels up to a length of 256 can easily be generated.

ii) Other useful methods

In order to obtain the different possible structures, Edwards' rules were introduced in an algorithm. However, the outcome was not specific enough, and the total number of structures did not follow the Wedderburn–Etherington sequences. Other methods were therefore used to obtain the intended result.

- In the fifth rule of Table 4, Edwards (1991) states "The sum of the digits in the first half of a label is greater than or equal to the sum of the digits in the second half of a label. This rule also applies to each quarter, eighth, etc., that is, the sum of the digits in the first quarter of a label is greater than or equal to the sum of the digits in the second quarter of a label and the sum of the digits in the third quarter of a label is **greater than** the sum of the digits in the fourth quarter of a label."

  It will be specified that the digits in the third quarter of a label must be **greater than or equal to** the sum of the digits in the fourth quarter of a label**.** As an example, for the label 21101010, the sum of the digits in the third quarter is equal to the sum of the digits in the fourth quarter.

- An additional rule, required in order to avoid redundant structures, specifies that the fourth quarter of a label cannot be exclusively composed of 0s. As an example, without that rule, a seven-player tournament with five rounds produces these labels:
    - 21101000010001000
    - 21101000010100000

  In both cases, the left halves are exactly the same. In the right halves, even though the real players are not located in the same leaves, the outcome will be the same. Indeed, these two right halves labels, although different in the tournament format (0-0-0-1 for first label vs 0-0-1-0 for the second), represent the same structure.

  Figure 28 shows the right halves of these two labels.

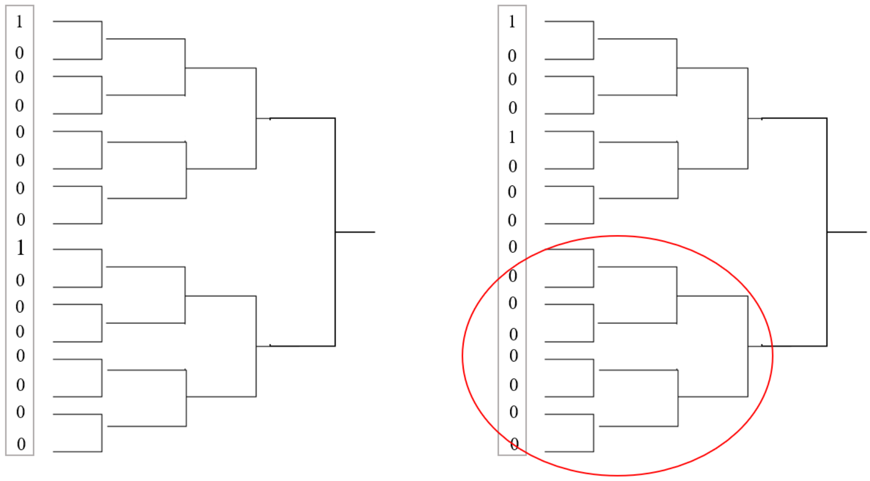

Figure 28: Label "21101000010001000" and "21101000010100000"

The label "21101000010001000" is preferred, as it does not involve a quarter composed of only 0's.

The only exception to this rule will be in the case of a totally unbalanced structure. In that case, as illustrated in Figure 29, the fourth quarter is fully filled with 0s but is nevertheless accepted.

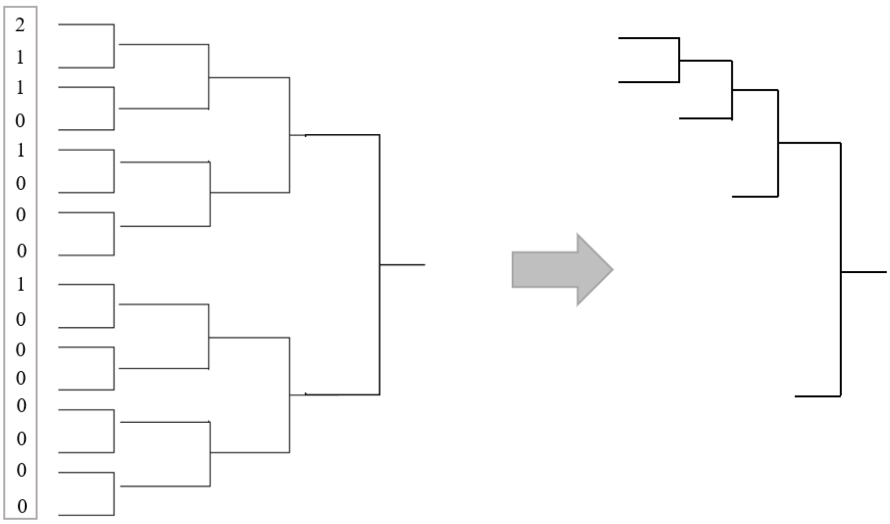

Figure 29: Totally unbalanced structure "2110100010000000"

In order to verify that we have the right number of structures for a p players tournament with r rounds, we use, once again, the formulas provided by Edwards (1991):

$$N(p,r) \;=\; \sum_{i\,=\,l}^{u} [N(p-i, r-1)]\,[N(i,r-1)\;+\;\delta_{p-i,i}]\,[1-\frac{\delta_{p-i,i}}{2}]$$

$$with\, N(p,r) \;=\; 1\, if\; p \leq\; 3$$

Therefore, for a tournament with p players (or t teams) and **at most** r rounds, we have:

	Values of r								
Values of t	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	1
4	1	2	2	2	2	2	2	2	2
5		2	3	3	3	3	3	3	3
6		2	5	6	6	6	6	6	6
7		1	6	10	11	11	11	11	11
8		1	8	17	22	23	23	23	23
9			8	25	39	45	46	46	46
10			9	38	70	90	97	98	98
11			7	52	118	171	198	206	207
12			7	73	200	325	406	441	450
13			4	93	324	598	811	928	972
14			3	121	526	1,097	1,613	1,951	2,113
15			1	143	825	1,972	3,155	4,046	4,555
16			1	172	1,290	3,531	6,141	8,349	9,795

Table 5: The number of tournament structures with t teams and at most r rounds. Reprinted from The combinatorial theory of single-elimination tournaments, by Christopher Edwards (1991)

In order to obtain the number of different structures for exactly r rounds, just subtract the left adjacent columns. The outcome is shown at Table 6.

	Values of *r*								
Values of *t*	2	3	4	5	6	7	8	9	10
3	1								
4	1	1							
5		2	1						
6		2	3	1					
7		1	5	4	1				
8		1	7	9	5	1			
9			8	17	14	6	1		
10			9	29	32	20	7	1	
11			7	45	66	53	27	8	1
12			7	66	127	125	81	35	9
13			4	89	231	274	213	117	44
14			3	118	405	571	516	338	162
15			1	142	682	1,147	1,183	891	509
16			1	171	1,118	2,241	2,610	2,208	1,446

Table 6: The number of tournament structures with t teams and exactly r rounds. Reprinted from the combinatorial theory of single-elimination tournaments, by Christopher Edwards (1991)

For confirmation, the sum of each line represents the Wedderburn–Etherington sequence.

3.2.2.2 Assignment of strengths and placement in the bracket

Once all the possible structures have been obtained, the players can be placed in the bracket.

As previously stated, this study used a perfect binary tree—that is, a binary tree in which all the external nodes are at the same level, to represent each possible structure. As mentioned in section 3.2.2, dummy players are added to fulfil a complete binary tree into a perfect binary tree (see Figure 23). Hence, a tournament of R rounds has p, the number of real players, and D, the number of dummy players, with the relation:

$$2^R = p + D$$

Each player of this set has to be assigned a strength, and these will be stored in a vector of size 2^R. Assigning a strength allow players to be ranked to determine the different odds of victory.

In this study, it was arbitrarily decided to represent these strengths by a random integer between 0 and 20, 0 being the lowest possible strength and 20 being the highest. All dummy players received a zero strength.

To know which players of the set will be dummy players, a function is used that translates the structure label, composed of digits 2, 1, and 0, into a sequence composed only of 1 and 0, which represents each external node of the perfect binary tree. An external node, represented by the digit 1, contains a real player, while those represented by the digit 0 contain dummy players.

This sequence was then multiplied by the vector containing the players' strengths. Thus, a dummy player has a strength equal to zero. To make this clearer, here is an example.

2210 is a possible five-player tournament structure. If it is translated in terms of external nodes, it produces the sequence 11111000, as shown in Figure 30, where P means that a real player is assign to that node, and D, in red, represents a dummy player.

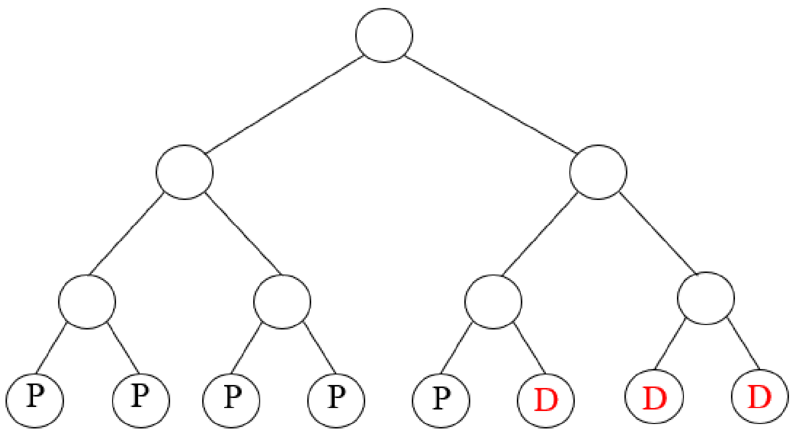

Figure 30: Representation of the "11111000" external nodes sequence

When this sequence is multiplied by a random strength vector, as shown in Figure 31, it produces the final player strength.

Figure 31: Method to obtain the final player strengths vector

This final player strengths vector represents one possible draw for a range of strengths. However, there is a multitude of different possible draws, which is why permutations will be done within the real players—that is, non-zero strengths (see Figure 32).

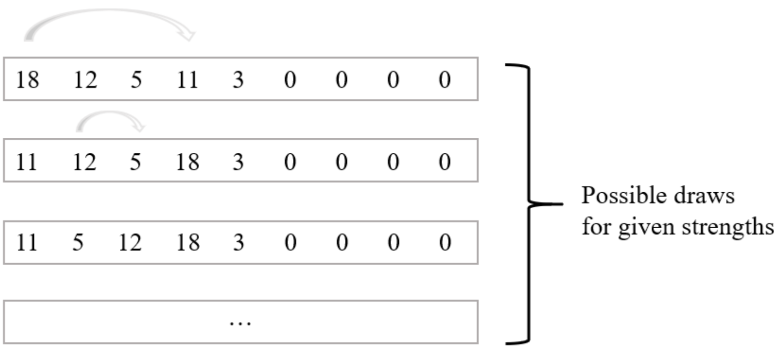

Figure 32: Permutations within final player strengths vector

All these final strengths vectors represent the possible draws.

3.2.2.3 Computation of the probabilities

As explained in Chapter 2, the probability W_{ir}—the probability that player i wins the round r is used to obtain W_{iR} —the probability that player i wins the last round, R, and thus, the tournament.

$$W_{ir} = W_{i,r-1}\left[\sum_{k=v}^{u} P_{ik} W_{k,r-1}\right] \; where \; W_{i,0} = 1 \; and \; r > 0$$

The first step is therefore to compute the strongest player's average probability of winning for all the possible draws of given strengths. The second step is to calculate it for a multitude of different strengths. Finally, the outcome is the strongest player's average probability of winning for a given structure.

3.2.2.4 Optimal structure

For each possible tournament structure of p players, the probability that a player i wins the tournament can be computed. Therefore, by taking the strongest player's probability of winning in each structure and by choosing the maximal one, the optimal type of structure can be deduced. In the same way, choosing the minimal probability reveals the structure that minimizes the strongest player's probability of winning.

3.3 RESULTS

3.3.1 All possible structures algorithm

i) Strongest player's probability

This section presents the results obtained via the algorithm for tournaments of up to 12 players. As previously mentioned, the draw can have a significant impact on the outcome of a tournament. For example, if the algorithm was run only once, and if the random draw resulted in the strongest player facing the second-strongest player in the first match of the first round, the probability would have been biased by an outstanding result. Therefore, in order to produce valuable and meaningful results, 100 permutations have been done into the final strengths vector (see Figure 32) to produce different draws with the same given strengths and the algorithm was run over 1000 iterations.

As can be seen in Figure 33, which shows the distribution of the average strongest player's probability of winning for the structure 2111, the results are stable, with a slight standard deviation (see Table 7).

Standard deviation (s)	0.002

Table 7: Standard deviation for 2111 structure

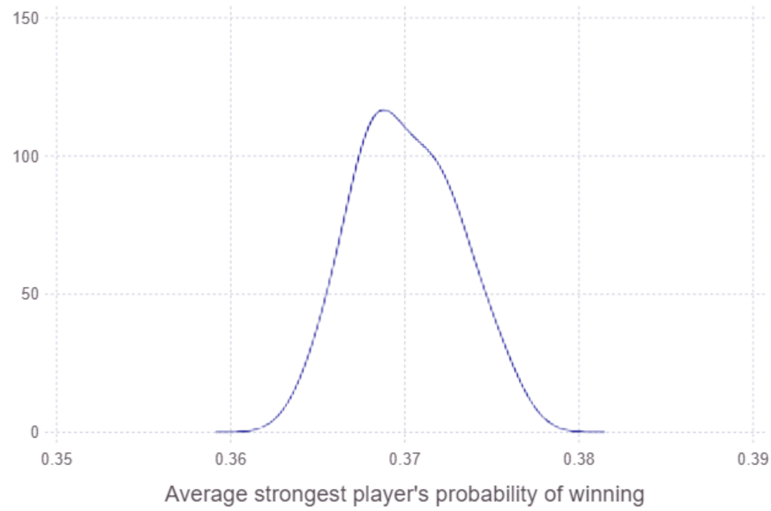

Figure 33: Probability distribution for 2111 structure

In the tables that follow, the left column represents the possible structures, and the right column represents the average strongest player's probability of winning. The lines corresponding to balanced structures are highlighted in grey.

- Two players

2	0.677

Table 8: Strongest player's probability of winning in a two-player tournament

- Three players

21	0.518

Table 9: Strongest player's probability of winning in a three-player tournament

- Four players

22	0.440
2110	0.412

Table 10: Strongest player's probability of winning in a five-player tournament

- Five players

2111	0.372
2210	0.355
21101000	0.344

Table 11: Strongest player's probability of winning in a five-player tournament

- Six players

2121	0.329
2211	0.326
22101000	0.296
21111000	0.306
21101010	0.315
2110100010000000	0.289

Table 12: Strongest player's probability of winning in a six-player tournament

- Seven players

2221	0.294
22111000	0.266
21211000	0.270
21111010	0.283
21101110	0.285

22101010	0.275
2111100010000000	0.255
2110101010000000	0.260
2210100010000000	0.273
2110100010001000	0.251
21101000100000001000000000000000	0.249

Table 13: Strongest player's probability of winning in a seven-player tournament

As revealed in Tables 8 to 13, the number of tournament structures and the length of the labels increases quickly. For more than eight rounds (i.e., label length 12) the probability calculation becomes computationally expensive. However, as can be observed, as the length of the label increases, the strongest player's probability of winning decreases. It can therefore be assumed that by only computing the "short" labels, the optimal one will become apparent.

Without showing all the different labels, the optimal structures for tournaments with eight players or more are as follows:

- Eight players

2222	0.271
21101111	0.265
21111110	0.259
21102110	0.259
21211010	0.253
22211000	0.239
22111010	0.251
22101110	0.255

Table 14: Strongest player's probability of winning in an eight-player tournament

- Nine players

21112110	0.240
21111111	0.243

22102110	0.236
22101111	0.240
21211110	0.237
22111110	0.235
22211010	0.229
22221000	0.221

Table 15: Strongest player's probability of winning in a nine-player tournament

- Ten players

21112111	0.226
22102210	0.219
22102111	0.222
21212110	0.220
21211111	0.223
22112110	0.219
22111111	0.222
22211110	0.216
22221010	0.211

Table 16: Strongest player's probability of winning in a ten-player tournament

- Eleven players

21212111	0.209
21212210	0.207
22112111	0.208
22112210	0.206
22212110	0.203
22211111	0.205
22221110	0.201

Table 17: Strongest player's probability of winning in an eleven-player tournament

- Twelve players

21212121	0.199
22112211	0.196
22112121	0.197
22212210	0.193
22212111	0.194
22222110	0.190
22221111	0.192

Table 18: Strongest player's probability of winning in a twelve-player tournament

As can be observed, for each tournament, it is the balanced structure that gives the highest winning probability for the strongest player. Indeed, for a tournament with p = $2^R + k$ players, the structure owing k matches in the first round followed by 2^R in the second round maximizes the strongest player's probability of winning the tournament.

However, as previously computed in Table 2, using Maurer's formula B_p, for some tournaments, different balanced structures can appear. For instance, when the number of players is equal to six, there are two possible balanced structures (see Figure 34).

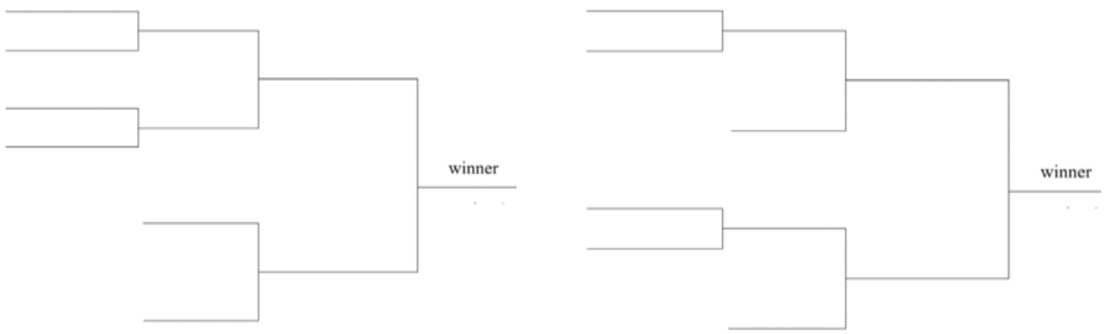

Figure 34: Possible balanced structure for a six-player tournament

In the left structure of Figure 34, the two first round matches are both in the upper part of the structure, while in the right structure, the two first round matches are separated, one in the upper part and one in the lower part of the structure.

As a result of the computed probabilities, it can be said that the right-hand structure gives a slightly higher probability of winning. This conclusion, although formulated differently, was also reached by Adler et al. (2017). Indeed, in their remark 6, they stated “In the case of general v_i, the format that calls for n/2 matches when an even number n of players remain is not optimal for the best player” (Adler et al., 2017, p. 7). The format they were talking about, for a six-player tournament, with three matches in the first round, is represented in Figure 35. As can be observed, the outcome is exactly the same as that illustrated in the left hand-side structure of Figure 34—whether the bottom bracket match is played in the first round or the second round, does not affect the players' probabilities of winning.

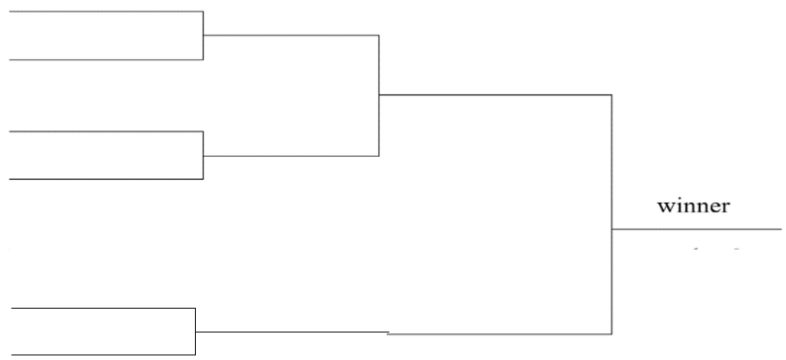

Figure 35: 3-1-1 format

Therefore, for a tournament with p = $2^R + k$ players, in order to obtain the optimal structure for the strongest player, the k matches played in the first round should be equally set out in the structure. If k is even, k/2 matches should be placed in the upper substructure and k/2 matches in the lower substructures. When k is odd, $\lfloor k/2 \rfloor + 1$ matches should be set out in the upper substructures and $\lfloor k/2 \rfloor$ matches in the lower substructures, or conversely.

In parallel, by taking the same tables, it can be observed that the strongest player’s probability of losing is increased with the totally unbalanced structure (i.e., only one match per round is played), which was also conjectured by Adler et al. (2017).

ii) Weakest player’s probability

Using the same method as was applied to the strongest player, the corresponding analysis can be done with the weakest player.

As demonstrated by Adler et al. (2017) for a particular case where $v_1 = v_2 = v_3 = \ldots = v_{p-1} > v_p$, where all players have the same strength except one, the weakest, who has a lower strength, the structure that maximizes the chances of victory for that player is the one where one match per round is played. The authors also proved that if $p = 2^R + k, 1 \leq k \leq 2^R$, the structure that minimizes the weakest player's chances of winning is the one in which k matches are played in the first round and then all remaining players compete in each following round (Adler et al., 2017).

As stated previously, the probability computation becomes more expensive as the number of rounds increases. Hence, another method of calculating the probability has been put in place, but only for totally unbalanced structures. In these structures, two players are first randomly chosen to play against each other in the first round, and the winner then faces another random player, and so on until there is only one player left. This technique ensures that no dummy players are added, because all players are chosen randomly, round by round. The probability calculation is therefore much faster.

For general cases where players have different strengths, the following results are produced for tournaments with up to six players:

- Two players

2	0.381

Table 19: Weakest player's probability of winning in a two-player tournament

- Three players

21	0.145

Table 20: Weakest player's probability of winning in a three-player tournament

- Four players

22	0.0724
2110	0.0811

Table 21: Weakest player's probability of winning in a four-player tournament

- Five players

2111	0.0426
2210	0.0519
21101000	0.0551

Table 22: Weakest player's probability of winning in a five-player tournament

- Six players

2121	0.0251
2211	0.0258
22101000	0.0278
21111000	0.0367
21101010	0.0383
2110100010000000	0.0397

Table 23: Weakest player's probability of winning in a six-player tournament

The highest probability for each table is highlighted in grey and always corresponds to the totally unbalanced structure. In view of the results, we therefore support the conjecture of Adler et al. (2017).

3.3.2 Optimal label algorithm

In order to directly obtain the optimal label for the strongest player, one last algorithm was developed. This algorithm returns, for every p players, the optimal structure—the one that will maximize the chances of the strongest player to win a tournament.

To do this, the same recursion principle as used in the previous algorithm was used.

For a number of rounds inferior to 3, possible labels of length 2^{R-1} with R=$\lceil log_2 p \rceil$ were generated. The only changes made were that the label could only contain 2s or 1s. Indeed, as only a limited number of dummy players are introduced, in order to achieve a perfect binary tree, the label will never contain 0.

The 2s represent the pre-round matches, and the number of 2s in the optimal labels is therefore equal to k. This implies that the number of 1s will be equal to 2^R – k:

$$Recall: p = 2^R + k , 0 \leq k < 2^R$$

The only difficulty, therefore, is in knowing where to place the 2s.

By observation, it can be deduced that if the number of players is even, which also implies that k is null or even, the number of pre-round matches, i.e. 2s, must be distributed equally between the top and bottom table—that is, equally distributed between the left and right half of the label. On the other hand, when the number of players is odd, k is odd, and thus $\lfloor k/2 \rfloor + 1$ matches should be set out in the upper substructures (the left half of the label,) and $\lfloor k/2 \rfloor$ matches in the lower substructures (the right half of the label.)

When the number of round R is higher than or equal to 3, the same recursion principle as that used in the previous algorithm is used. Indeed, the left side of the label is an optimal label for a tournament with fewer players. The right side of the label is again an optimal label of a tournament with fewer players or a sequence only composed of 1s.

The output of all these rules gives a unique label representing the optimal tournament structure for p players (see Table 24).

Number of players, p	Optimal structure
2	2
3	21
4	22
5	2111
6	2121
7	2221
8	2222
9	21111111
10	21112111
11	21212111
12	21212121
13	22212121
14	22212221
15	22222221
16	22222222

Table 24: Optimal structure labels for up to twelve players

Chapter 4: Conclusion and Future Work

For many years, researchers have been interested in problems related to the design of sports tournaments. Their studies have dealt with such areas as choosing the best tournament type; the optimal way to build a draw, (i.e. where to place players in the tournament in order to optimize the winning probability of a given player); and the best way to rank players according to several criteria. However, relatively few studies have looked at the structure (i.e., the skeleton) of these tournaments, even though structure has a big impact on the outcome of the competition.

The only type of tournament structure that is regularly considered is the balanced structure, where there are p/2 matches in the first round, with p the number of players, in cases where the number of players is equal to a power of 2. In cases where the number of players is not a power of 2, k matches are played in the first round, with p = $2^R + k$ and where $0 \leq k < 2^R$, followed by the balanced structures. Hence, several authors have started to take an interest in the different types of possible structure—for example, Maurer (1975), Chung and Hwang (1978), and Edwards (1991). The most recent, Adler et al (2017), demonstrated that in the particular case where there is one strongest player and all the remaining players have the same strength, it is the balanced format that maximizes the strongest player's probability of winning in a single-elimination tournament. Adler et al (2017) also stated that this probability is minimized with a totally unbalanced structure (i.e., one match per round). The general case, where all players have different strengths, has been briefly analyzed by the authors and only assumptions have been made.

For this reason, the objective of this thesis was to study in depth the different types of tournament structure and to answer this research question:

"In a knockout tournament, what type of structure optimizes the strongest player's probability of winning?"

The aim was therefore to generate all the different possible structures of a p players knockout tournament and to find the optimal one. As the number of possibilities increases sharply with the number of players, an algorithm was developed in order to determine the different structures, and to compute the associated probability of winning for the strongest player.

From the results obtained—presented in the tables in section 3.3—it is possible to conclude that it is effectively the balanced structure that maximizes the strongest player's probability of winning in the case of random knockout tournaments. We therefore support the conjecture of Adler et al. (2017).

Furthermore, for a tournament with p = $2^R + k$ players, the k matches played in the first round should be equally set out in the structure to maximize the strongest player's probability of winning. If k is even, k/2 matches should be placed in the upper substructure and k/2 matches in the lower substructures. When k is odd, $\lfloor k/2 \rfloor + 1$ matches should be set out in the upper substructures and $\lfloor k/2 \rfloor$ matches in the lower substructures.

In parallel, the different structures that minimize the strongest player probability were similarly analyzed. Thus, we can also support the conjecture of Adler et al. (2017) saying that it is the totally unbalanced structure (i.e., the one match per round structure) that minimizes the strongest player's probability of winning.

Finally, using the same approach as that used to assess the strongest player's probability of winning, the weakest player's probabilities were evaluated. As Adler et al. (2017) assumed in their paper for the particular case they studied, we also conclude, even in more general cases, that the weakest player's probability of winning is maximized under the totally unbalanced structure and minimized under the balanced structure with k matches in the first round.

LIMITATIONS AND FUTURE WORK

The algorithm proposed in section 3.2.3, based on Edwards' rules, provides, for tournaments with up to 10 players, the exact number of tournament structure. However, beyond ten players, some structures seem to be missing. Indeed, the number of structures for some rounds is not reached. As may be noticed at Table 6, for a tournament of eleven players and four rounds, 45 different structures should be found. However, our algorithm comes up with only 43, giving a total of 205 different structures for an eleven-player tournament, instead of 207 as the 11^{th} number in the Wedderburn–Etherington sequence.

The missing structures never being the balanced ones and due to the fact that they appear only from eleven players and only for certain rounds, our results should not be affected. Indeed, it is unlikely that one of the missing structures returns a higher winning probability for the strongest player.

Nevertheless, one possible avenue for future work would be to work on the algorithm to check why, at times, structures are missing. Moreover, the algorithm could be improved, for example by reducing the complexity, or finding a possible alternative method. Indeed, as the proposed algorithm includes nested loops for skimming the different structures labels, the complexity is important. Hence, an algorithm that directly gives the accepted structures according to the different rules without first having to generate all the possible labels could be considered.

Additionally, regarding the algorithm, the suggested probability calculations become computationally expensive as the number of rounds increases. Other methods, not including each round probability computation, should therefore be examined.

Concerning the starting point of this thesis, although we were able to support the conjecture of Adler et al. (2017), we could not prove that it is indeed the balanced structures that maximize the chances of victory of the strongest player and the totally unbalanced structures that minimize those chances. A mathematical demonstration would therefore be valuable.

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Executive Summary

For many years, researchers have investigated problems related to the design of sports tournaments. Sports competitions involve many logistical and economic issues, which has led many authors to examine them from a more theoretical point of view. Many studies deal with the best tournament type choice, and the optimal way to devise a draw—that is, where to place players in the tournament in order to optimize the winning probability of a given player, deciding the best way to rank players according to several criteria, and other issues. However, there are relatively few studies on the structure (i.e., the skeleton) of these tournaments, although structure has a big impact on the outcome of the competition.

The purpose of this thesis is, therefore, to analyze the different tournament structures and to infer which ones maximize or minimize the strongest player's probability of winning. The research question of this dissertation is: "In a knockout tournament, that is to say, direct elimination tournament, what type of structure optimizes the strongest player's probability of winning?"

During the elaboration of this paper, different sports tournaments and their specific terminology are explained, winning probabilities of random knockout tournament are computed, and an algorithm is developed in order to provide indications of the effectiveness of the tournament structure and to evaluate and draw conclusions on the types of structure to be chosen.

As a result, we support the conjectures of Adler et al. (2017), saying that, in a random knockout tournament and in a general case where the players all have different strengths, the balanced structures maximize the chances of victory for the strongest player. In addition, we also achieve that the structures minimizing the winning probability of the strongest player, are the totally unbalanced ones, that is to say, those where only one match per round is played. Concerning the weakest player, the same analyses were carried out and it was concluded, as Adler et al. (2017), that, conversely, balanced tournaments minimize the chances of victory of the weakest player and totally unbalanced structures maximize them.